Secret Identities

Overview

Description

Pattern recognition and symbolic representation of relationships are fundamental skills in mathematics, but sophisticated patterns can be difficult to discern and write down. This assignment challenges students to determine identities that govern given number patterns, and then express the identities algebraically.

Final Product: Students will discover algebraic identities through pattern recognition from a small sample of specific instances of the identities. They will represent their identities as equations in one or two variables and create a "challenge pattern" for the rest of the class. They will attempt to determine the identity of other students in class and then write up their solutions.

Subject

Algebra II

Task Level

Grade 10-12

Objectives

Students will:

- Determine the strategies needed to solve the problems by interpreting the given information.
- Solve problems whose solutions involve creating, transforming, and analyzing algebraic expressions and equations.
- Discover identities and create examples of them so that other students can discover them as well.

Preparation

- Read the Instructor Task Information and the Student Notes.

Prior Knowledge

Students should be able to formulate and manipulate algebraic expressions and equations using algebraic (field) properties, concepts, procedures, and algorithms to combine, transform, and evaluate expressions (e.g., polynomials, rational expressions).

Key Concepts and Terms

- Identity
- Variable

Time Frame

Plan 40 minutes for students to solve the problems individually, 40 minutes to write up the solution, and 60 minutes for students to come up with algebraic identities and discover those that other students have created.

Instructional Plan

Getting Started

Learning Objectives

Students will:

- Determine the strategies needed to solve problems.
- Identify a general pattern from specific instances of an algebraic identity.
- Express those patterns in algebraic language.

Procedure

1. Display these three equations.

$$\frac{1}{2}-\frac{1}{3}=\frac{1}{6} \qquad \frac{1}{4}-\frac{1}{5}=\frac{1}{20} \qquad \frac{1}{9}-\frac{1}{10}=\frac{1}{90}$$

2. Ask students to look for a pattern that is true for all three equations. When students are ready, ask for their verbal descriptions of the pattern. Ask students to agree on a clearly stated description. If students have trouble identifying a relationship, lead them through the questions below as far as necessary until they see how to proceed.
   - What can you say about the numerators?
   - What is the relationship between the denominators on the left-hand side
   - How are those denominators related to the denominator on the right-hand side?
3. Ask students to use an algebraic equation in one variable to represent the pattern.
4. If students are stuck, suggest letting n be the value of the first denominator in each equation, and ask them what the value of the second and third denominators should be.
5. The identity can be described by the equation

   $$(1/n)-1/(n+1) = 1/(n(n+1))$$

   Once students have derived this, ask them to verify the identity by simplifying the left-hand side and comparing the result to the right-hand side.
6. Ask, Is the equation true for all values of n?

   Discuss why $n \neq -1$ or 0. Note the use of field properties.

Investigating

Learning Objectives

Students will:

- Determine the strategies needed to solve the problems by interpreting the given information.
- Solve problems whose solutions involve creating, transforming, and analyzing algebraic expressions and equations.

Procedure

1. Students individually find an identity for each set of three equations on the student handout.

Drawing Conclusions

Learning Objectives

Students will:

- Discover an algebraic identity.
- Use three examples of their partner's algebraic identity to discover the partner's identity.
- Communicate their findings.

Procedure

1. Students follow the instructions in the Student Notes to find an algebraic identity.
2. Students create three examples of their identity, write them on a separate sheet of paper, and trade with a partner, keeping their identities secret.
3. Using their partner's examples, students attempt to determine their partner's secret identity.
4. Students write up their solutions, each step clearly so that another student could easily follow the steps and come to the solution the same way.
5. You might end the assignment by having students share their problems for the class to solve (with partners abstaining). Have the class vote for the most interesting identity, the most elusive identity, the fastest identity "spotter," and so on.

Scaffolding/Instructional Support

The goal of scaffolding is to provide support to encourage student success, independence, and self-management. Instructors can use these suggestions, in part or all together, to meet diverse student needs. The more skilled the student, however, the less scaffolding that he or she will need. Some examples of scaffolding that could apply to this assignment include:

- Encourage students to share their ideas in small groups.
- Allow students to work individually, with a partner, or in small groups. Pair or group students with a more advanced understanding with students whose understanding is less advanced.
- Check in with students at regular intervals during the assignment to ensure that they are progressing in a productive manner. Redirect and clarify any of the types of questions for the students, if needed.
- Ask students who are having trouble with the written elements of the assignment to explain their answers verbally, and then assist them in translating verbal expression to written expression.

Solutions

The solutions provided in this section are intended to clarify the problems for instructors. The solutions may not represent all possible strategies for approaching the problems or all possible solutions. Solutions are provided for reference only.

a. Let n be the smaller number. Then $\frac{1}{n} - \frac{1}{n+2} = \frac{2}{n(n+2)}$.

Since $\frac{(n+2)-n}{n(n+2)} = \frac{2}{n(n+2)}$, the statement is true for all values of $n \neq -2$ or 0.

b. Let x be the first number: $x^2 + (1 - x) = x + (1 - x)^2$

The statement is true for all values of x when $0 \leq x \leq 1$.

c. Let n be the smallest number:

$$n^2 + (n+1)^2 = [n(n+1)+1]^2 - [n(n+1)]^2$$

The statement for all values of n can be verified in two ways:

$$n^2 + (n+1)^2 = [n(n+1)]^2 + 2n(n+1) + 1 - [n(n+1)]^2 = 2n^2 + 2n + 1$$

or factor the right side using the difference of squares

$$[n(n+1)+1+n(n+1)][n(n+1)+1-n(n+1)] = [2n^2+2n+1][1] = 2n^2+2n+1$$

d. Let x be the first number and let y be the second number. Then

$$(x^2 + 1)(y^2 + 1) = (xy + 1)^2 + (x - y)^2$$

This can be verified by multiplying out both sides. Here, both sides are equal to

$$x^2y^2 + x^2 + y^2 + 1$$

TCCRS Cross-Disciplinary Standards Addressed

Performance Expectation	**Getting Started**	**Investigating**	**Drawing Conclusions**
I. Key Cognitive Skills			
A.1. Engage in scholarly inquiry and dialogue.	✓	✓	✓
B.1. Consider arguments and conclusions of self and others	✓		
B.4. Support or modify claims based on the results of an inquiry.	✓	✓	✓
C.2. Develop and apply multiple strategies to solve problems.		✓	
D.1. Self-monitor learning needs and seek assistance when needed.		✓	✓
D.3. Strive for accuracy and precision.		✓	
D.4. Persevere to complete and master tasks.			✓
E.1. Work independently.	✓	✓	✓
II. Foundational Skills			
B.1. Write clearly and coherently using standard writing conventions.			✓

TCCRS Mathematics Standards Addressed

Performance Expectation	**Getting Started**	**Investigating**	**Drawing Conclusions**
II. Algebraic Reasoning			
B.1. Recognize and use algebraic (field) properties, concepts, procedures, and algorithms to combine, transform, and evaluate expressions (e.g., polynomials, radicals, rational expressions).	✓		
C.2. Translate among multiple representations of equations and relationships.		✓	
D.1. Interpret multiple representations of equations and relationships.			✓
VIII. Problem Solving and Reasoning			
A.1. Analyze given information	✓		

A.2. Formulate a plan or strategy	✓		
A.3. Determine a solution.		✓	
A.4. Justify the solution.			✓
IX. Communication and Representation			
A.1. Use mathematical symbols, terminology, and notation to represent given and unknown information in a problem.	✓		
A.2. Use mathematical language to represent and communicate the mathematical concepts in a problem.		✓	
A.3. Use mathematics as a language for reasoning, problem solving, making connections, and generalizing.		✓	
B.1. Model and interpret mathematical ideas and concepts using multiple representations.			✓
C.1. Communicate mathematical ideas, reasoning, and their implications using symbols, diagrams, graphs, and words.			✓

TEKS Standards Addressed

***Secret Identities** - Texas Essential Knowledge and Skills (TEKS): Math*
111.32.b.1. Foundations for functions. The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways. The student is expected to: 111.32.b.1.D. represent relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities.
111.32.b.3. Foundations for functions. The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations. The student is expected to: 111.32.b.3.A. use symbols to represent unknowns and variables. 111.32.b.3.B. look for patterns and represent generalizations algebraically.
111.32.b.4. Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations. The student is expected to: 111.32.b.4.A. find specific function values, simplify polynomial expressions, transform and solve equations, and factor as necessary in problem situations. 111.32.b.4.B. use the commutative, associative, and distributive properties to simplify algebraic expressions.

Secret Identities - *Texas Essential Knowledge and Skills (TEKS): Math*
111.32.b.5. Linear functions. The student understands that linear functions can be represented in different ways and translates among their various representations. The student is expected to: 111.32.b.5.A. determine whether or not given situations can be represented by linear functions.
111.32.b.7. Linear functions. The student formulates equations and inequalities based on linear functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation. The student is expected to: 111.32.b.7.A. analyze situations involving linear functions and formulate linear equations or inequalities to solve problems.
111.33.b.8. Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation. The student is expected to: 111.33.b.8.A. analyze situations involving quadratic functions and formulate quadratic equations or inequalities to solve problems.
111.34.b.5. Geometric patterns. The student uses a variety of representations to describe geometric relationships and solve problems. The student is expected to: 111.34.b.5.A. use numeric and geometric patterns to develop algebraic expressions representing geometric properties.
111.36.c.1. The student uses a variety of strategies and approaches to solve both routine and non-routine problems. The student is expected to: 111.36.c.1.B. use multiple approaches (algebraic, graphical, and geometric methods) to solve problems from a variety of disciplines. 111.36.c.1.C. select a method to solve a problem, defend the method, and justify the reasonableness of the results.

Secret Identities

Introduction

You will consider four sets of three equations each to find an algebraic identity that leads to each specific equation. You will express the algebraic identity as an equation in one variable, and determine all values of the variable for which it holds.

Directions

Getting Started

1. Your instructor will introduce the activity.
2. The class will work one problem together.

Investigating

1. Now solve some similar problems on your own. Below are three groups of equations. For each group:
   - Look for a number pattern.
   - Write an equation in one variable that represents the pattern.

a.	b.	c.
$\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$	$(0.45)^2 + 0.55 = 0.45 + (0.55)^2$	$2^2 + 3^2 = 7^2 - 6^2$
$\frac{1}{4} - \frac{1}{6} = \frac{2}{24}$	$(0.23)^2 + 0.77 = 0.23 + (0.77)^2$	$3^2 + 4^2 = 13^2 - 12^2$
$\frac{1}{9} - \frac{1}{11} = \frac{2}{99}$	$\left(\frac{1}{3}\right)^2 + \frac{2}{3} = \frac{1}{3} + \left(\frac{2}{3}\right)^2$	$5^2 + 6^2 = 31^2 - 30^2$

2. The next set of equations comes from an identity in *two* variables. Look for a number pattern in the given equations and use it to determine such an identity.

$$(5^2+1)(3^2+1) = 16^2+2^2$$

$$(7^2+1)(4^2+1) = 29^2+3^2$$

$$(13^2+1)(6^2+1) = 79^2+7^2$$

$$(4^2+1)(6^2+1) = 25^2+2^2$$

Drawing Conclusions

1. For each set of equations you've worked with, give a written explanation of the pattern and the equation representing it. Your write-up should include algebraic verification of each identity.
2. Use the guidelines below to discover your own algebraic identity.
   a. Your identity should involve only one variable.
   b. Here is one way to derive an identity:

      Starting with $n = n$, manipulate both sides of the equation *without making them unequal* until they look very different from each other.

      For example, the first step could be to add $n^2 + n + 1$ to both sides, giving

      $$n^2 + 2n + 1 = n^2 + 2n + 1$$

      Factoring the right-hand side and rearranging the left-hand side would make this

      $$(n + 1)^2 = 2(n^2 + n) - n^2+1$$

   c. Be creative, but make sure the two sides remain equal! Any legal operation is fair game. You are not restricted to this procedure; if you have another approach to finding an identity, take it.

3. Without telling anyone what your identity is, create three numerical examples that illustrate it, similar to the groups of equations in *Getting Started* and *Investigating*. Write your examples on a separate sheet of paper. In the example above, taking $n = 2$ would give

$$3^2 = 12-4+1$$

   Simplify each example enough to disguise the identity but not so much that there is no information left to work with. In the example, simplifying to $9 = 8 + 1$ would be too much; you need to preserve the structure of your identity while disguising its particulars.

4. Still keeping your identity secret, give your equations to your partner. While your partner is trying to learn your secret identity, try to learn his or hers.

5. Write up your solution to your partner's secret identity problem so that another student in the class could recreate your work.