Let the Games Begin

Overview

Description

Students will apply their understanding of probability, fairness, and area and will discover how these concepts are related. Students will evaluate fairness associated with various games of probability and explore how changing the parameters of a game will affect its fairness. This task will include simulating events to gather data, comparing experimental and theoretical probabilities, and calculating expected values to determine the fairness associated with various games of probability.

Final Product: Working in small design teams, students will apply their understanding of probability and expected value to create a game of probability that satisfies a set of criteria from a company interested in investing in new games. Teams will provide an analysis of their game that includes experimental and theoretical probabilities and expected value in order as part of a proposal to convince the company to invest. Finally, teams will present the games to classmates who will play the role of the company, evaluating the games and deciding to accept or deny each of the proposals.

Subject

Geometry or Mathematical Models with Applications

Task Level

Grades 9–11

Objectives

Students will:

- Calculate experimental and theoretical probabilities and explain any differences between the two.
- Find and list all possible outcomes in an organized manner.
- Analyze and calculate the probability of a situation.
- Use probability to make predictions.
- Analyze the components of a probability game and construct the necessary parameters to produce a fair game.
- Associate expected value as a long-term average and explain how to calculate this value.

- Design a game of probability that adheres to a given set of criteria and provide an analysis of the game.
- Design and conduct simulations to gather and record experimental data.
- Extend their knowledge of expected value and apply strategies to projecting expected profit of a game.

Preparation

- Read the Instructor Task Information and the Student Notes.
- Prepare student copies of the Student Notes and handouts.
- Gather the following materials: paper clips or bobby pins to simulate spinners (at least two paper clips for each pair of students), map pencils, large gridded chart paper and dot labels for combined class data (optional), and graphing calculators.
- Have a variety of materials available that design teams may use to create their own game of probability in the Drawing Conclusions portion of this task. Materials may include, but are not limited to:
  - Pre-made spinners of various sizes (10-section, 12-section, etc.) or compasses and protractors to create a custom spinner.
  - Paper clips.
  - Colored counters, cards, colored paper, markers, tape, glue.
  - Coins, number cubes, calculator/computer with a random number generator.
  - Large chart paper or grid paper for area models.

Prior Knowledge

- Students should know the difference between experimental and theoretical probability. Students should be able to find all the outcomes of a multiple-stage event in an organized manner. Students will also be expected to be able to calculate theoretical probability and compare it to experimental probability.
- Students should be able to calculate expected values.
- Students should be comfortable with calculating the mean.
- Students should know how to work with equivalent ratios in fraction, decimal, and percent form.
- Students should be able to perform all operations with rational numbers.
- Students will need to recall the effects of scale factor on area, use a protractor to create a circle, and calculate a portion of the area of a circle.

Key Concepts and Terms

- Area model
- Complement of an event
- Equally likely
- Event
- Expected value
- Experiment
- Experimental probability
- Fairness
- Law of Large Numbers (LLN)
- Outcome
- Prediction
- Probability
- Sample space
- Theoretical probability
- Trial

Time Frame

This assignment will take approximately one week. Portions of this task can be assigned and completed outside of class at the discretion of the instructor.

Instructional Plan

Getting Started

Learning Objectives

Students will:

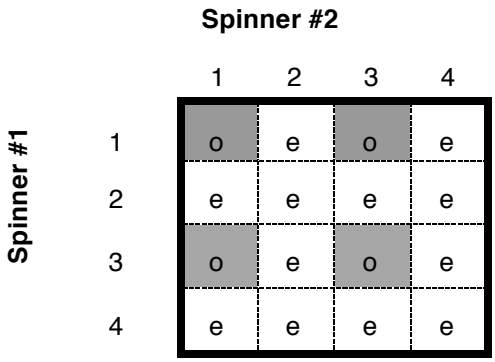
- Analyze and calculate the probability of a situation and illustrate the probability using an area model.
- Associate expected value as a long-term average and explain how to correctly calculate this value.
- Use probability to make predictions.

Procedure

1. Distribute the Student Notes. Focus students' attention on Getting Started. Explain that the objective is to analyze a game and determine whether it is a fair game. Facilitate a class discussion of students' thinking about games they have played. Elicit ideas about the following.
   - *Was the game fair?*
   - *What is the meaning of a "fair" game?*
   - *Was the game a game of chance (game of probability) such as randomly picking a card, or a game of skill such as aiming for a target, like darts?*
2. Model a few rounds of the game *Even or Odd?* with the whole class and demonstrate how to keep track of the score. Note the rules handouts.
3. To each pair of students, distribute the handouts, *Even or Odd?* and *Score Card—Even or Odd? Score,* along with two paper clips. Have students create spinners by slipping the end of a paper clip through the center of each spinner. Direct partners to play the game 25 times. Students should record the data properly. They will be used later in the analysis of the game.
4. When partners have completed the game, distribute *Game Analysis—Even and Odd?* and have partners complete it. Circulate and listen to students' responses. Look for students who have organized ways of listing all the outcomes, and plan to have those students share their methods with the class, as needed. Question students to ensure they understand the difference between experimental probability and theoretical probability, and make sure that they know how to calculate these probabilities.
5. Facilitate a class discussion of students' analysis of the game. Begin the discussion by asking the following.
   a. *Did you think the game was fair? Why or why not?*

b. If there were just as many even numbers as odd numbers on the spinners, and everyone received the same number of points, why wasn't the game fair?

6. Discuss strategies for listing all of the possible outcomes. Call on students who have interesting and organized strategies such as the one shown below. Elicit thinking about the question, *Why are there more even products than odd products possible?*



7. Discuss the remaining student responses to *Even or Odd? Game Analysis*. Review the terms random, equally likely, event, complement of an event, experimental probability, theoretical probability, and outcomes.

Investigating

Learning Objectives

Students will:

- Analyze the components of a game and construct the necessary parameters to produce a fair game of probability.
- Associate expected value as a long-term average and explain how to correctly calculate the value.
- Design and conduct simulations to gather and record experimental data.

Procedure

1. To develop understanding of *expected value*, facilitate discussion along the lines of the script that follows.

   a. For each round of play, either the "evens" player or the "odd" player would win 2 points and the opponent would receive 0 points.

   Look at your data and estimate the average of the number of points you won per round. How could we calculate this average?

   b. *Based on your experimental data, what was the mean number of points you won per round?*

(For example: If the "odd" player won 2 times in 25 trials for a total of 4 points then the mean would be 4 out of 25 or 0.16 points per round.)

What does this value mean in terms of the game?

c. *On average, based on your previous response, if you played the game 100 times, how many points could the 'even' player expect to win each round? The "odd" player?*

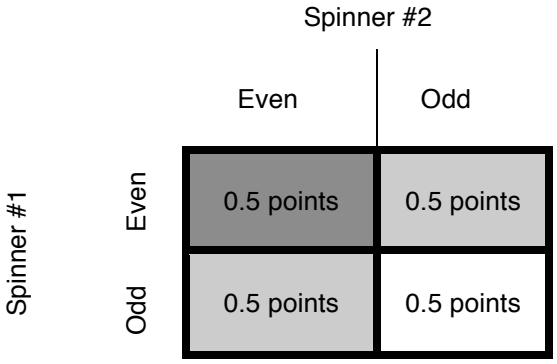
(For example: If the "odd" player wins an average of 0.16 points per round, then after 100 rounds he would be expected to have 100×0.16 or 16 points.)

d. *Explain how you could predict the average number of points per trial using the theoretical probabilities.*

(For example: If the "odd" player is expected to win 2 points 1/4 of the time, then it would be expected that he would win an average of 2 x 1/4 or 1 out of 2 points for each round.)

e. *Use the area model to illustrate how to calculate expected value.*

The table below shows the average points awarded per round for each combination of even and odd spins on the two spinners (see page SH-3).



Outcome	Probability of Winning	Points Won	Average Expected Points
Even	$\frac{3}{4}$	2	1.5
Odd	$\frac{1}{4}$	2	0.5

2. Introduce, define, and compare *long-term average* and *expected value.* Be sure students are comfortable with the terms and the calculations.
3. Ask students to consider the fairness of this game.

- *Do you think 'Even and Odd?' is a fair game? Explain.*
- *If a game is fair, what must true about the expected values?*
- *In this case, since the game is not fair, what changes could be made to the game to ensure it is a fair game?* There are many ways students can change the parameters of the game so it is fair. Some students may suggest altering the point system; others may suggest changing the areas of the spinners so that landing on an even or odd number is not equally likely. Another possibility is to change the rules of the game to find the sum of the numbers instead of the product. Be sure all ideas are discussed.
- *Would changing the numbers on the spinners make the game fair?*

Drawing Conclusions

Learning Objectives

Students will:

- Design a game of probability that adheres to a given set of criteria, and provide an analysis of the game.
- Design and conduct simulations to gather and record experimental data.
- Calculate experimental and theoretical probabilities and explain any differences between the two.
- Analyze and calculate the probability of a situation and illustrate this probability using an area model.
- Extend their knowledge of expected value and apply strategies to projecting expected profit of a game.

Procedure

1. Place the materials for making games in one area of the room so they are accessible to students for the duration of this task.
2. Launch the *Carnival-O-Rama* scenario as outlined in the Student Notes. Briefly discuss the scenario with students.
   - Emphasize that the carnival owner is looking for a game of probability, not skill, that will be attractive to customers and will be profitable for the carnival.
   - The owner of the carnival wants a game where the customer will be comfortable with the odds of winning, but the owner can expect a profit over time.
   - The owner knows that if the customer believes there is little probability in winning, the customer will not return to play the game.

3. Create design teams of three students.
4. Direct students' attention to the *Carnival-O-Rama Game Proposal Criteria*.
5. Show the materials teams may choose to use for. Explain that students may choose to use other materials, but prior approval by you is required.
6. Direct design teams to follow the directions outlined in the Drawing Conclusions section of the Student Notes. Give students a timeline and set a date for proposals.
7. When design teams are ready, have them present their games. Explain that the task of the class is to listen and assess each proposal. Distribute enough copies of *Carnival-O-Rama Proposal Criteria Peer Assessment* for pairs to complete the form for each proposal.

   Discuss the responsibility of completing a peer assessment fairly and critically. Stress to students that they must include a written comment explaining each score given. If time is a constraint, modify this portion of the task by having design teams present their proposal to only one or two other teams.
8. Completed peer assessment forms should be given to the presenting design teams for reflection. Consider allowing teams to revise their work based on the feedback given by peers. If a team disagrees with an assessment, ask them to submit their disagreement in writing for review.

Scaffolding/Instructional Support

The goal of scaffolding is to provide support to encourage student success, independence, and self-management. Instructors can use these suggestions, in part or all together, to meet diverse student needs. The more skilled the student, however, the less scaffolding that he or she will need. Some examples of scaffolding that could apply to this assignment include:

- Discuss the terms that students will be expected to know. Ask students to give examples, as needed.
- Discuss the rules for the game *Even or Odd?* and how to record the results.
- Allow students to work with a partner or in small groups. Group students with a more advanced understanding with students whose understanding is less advanced.
- Discuss how to illustrate the probability of an outcome using an area model when there are multiple events or stages in the game.
- Check in with students at regular intervals during all parts of the assignment to ensure that they are progressing in a productive manner. Redirect and clarify if necessary.

Solutions

The solutions provided in this section are intended to clarify the problem for instructors. These solutions may not represent all possible strategies for approaching the problem or all possible solutions. Solutions are provided for reference only.

Even or Odd?

1. Since all outcome pairs (even-even, even-odd, etc.) are equally likely and there are 3 possible events that will result in an even product, the probability of the "even" partner winning is 3 out of 4. Likewise, since there is only 1 opportunity for the "odd" partner to win, the probability that the "odd" partner will win is 1 out of 4.

   Since the odd and even outcomes are not equally likely to occur, the game cannot be considered fair. For the game to be fair there would have to be an equal number of opportunities for the product to be "even" as "odd."

2. The possible outcomes that can occur in one round for Spinners #1 and #2, along with their theoretical probabilities, are:

Event	Probability	Event	Probability
1x1	1/12	**1x2**	1/12
1x3	1/24	**1x4**	1/24
2x1	1/12	**2x2**	1/12
2x3	1/24	**2x4**	1/24
3x1	1/12	**3x2**	1/12
3x3	1/24	**3x4**	1/24
4x1	1/12	**4x2**	1/12
4x3	1/24	**4x4**	1/24

3. The probability of the product being even is the sum of the probabilities of the even products in the table above (the bold-face entries):

   $$(1/12+1/24+1/12+1/12+1/24+1/24+1/12+1/24+1/12+1/12+1/24+1/24) = 3/4$$

   The probability of the product being odd (the complementary event) is 1/4.

4. If *Even or Odds?* is played 100 times with spinner #1, then it is likely that the "even" player will win 75 times out of 100 and the "odd" player will win only 25 times out of 100.

   When spinner #2 is used, then it is likely that the "even" player will win approximately 72 times out of 100 and the "odd" player will win only 29 times out of 100.

5. Answers will vary.

Carnival-O-Rama

Solutions will vary depending on the student-made game. Below is a sample of what can be expected.

Aces Wild!

Rules:

A player rolls two dice. If the sum showing is exactly 5, he or she wins $5. If the sum is greater than 10 (i.e., 11 or 12), he or she wins $3. When an "ace" (one pip) appears, the player gets to choose any value for it, from 1 to 6. The game costs $2 to play.

I played the game 30 times. I won $5 seven times and $3 five times, for a total payout in 30 games of $50. At $2 per game, it cost me $60 to win the $50, so the net to the carnival would have been $10.

5	0	5	0	0
0	3	0	5	0
0	0	5	0	0
0	3	0	0	0
0	3	0	0	5
3	5	0	3	5

Results of Trials

The probability of winning a prize is 16/36 = 4/9; the probability of losing is therefore 5/9. This follows from the outcome table below, which also includes the payout for each winning dice roll. I considered a "win" to be anything except a 0.

	1	2	3	4	5	6
1	5	5	5	5	3	3
2	5	0	5	0	0	0
3	5	5	0	0	0	0
4	5	0	0	0	0	0
5	3	0	0	0	0	3
6	3	0	0	0	3	3

Table of Outcomes

The probability of winning $5 is 9/36=1/4, and the probability of winning $3 is 7/36. Since the games costs $2 to play, the expected value is

$$(1/4)(5) + (7/36)(3) - 2 = -0.1666\ldots \text{ dollars.}$$

That is, the carnival can expect to profit about 17 cents per game, on average.

In my simulation, I won 12 games out of 30, compared with an expected (16/36)(30) = 13.3, which is pretty close. I also lost $10 in 30 games, for an average loss of about 33 cents per game compared with the expected 17-cent loss per game. These seem to be in line with my calculated expected values.

TCCRS Cross-Disciplinary Standards Addressed

Performance Expectation	Getting Started	Investigating	Drawing Conclusions
I. Key Cognitive Skills			
A.1. Engage in scholarly inquiry and dialogue.	✓	✓	
A.2. Accept constructive criticism and revise personal views when valid evidence warrants.			✓
B.1. Consider arguments and conclusions of self and others			✓
B.2. Construct well-reasoned arguments to explain phenomena, validate conjectures, or support positions.			✓
B.3. Gather evidence to support arguments, findings, or lines of reasoning.		✓	✓
B.4. Support or modify claims based on the results of an inquiry.		✓	✓
C.1. Analyze a situation to identify a problem to be solved.	✓		
C.2. Develop and apply multiple strategies to solve problems.		✓	
C.3. Collect evidence and data systematically and directly relate to solving a problem.		✓	
D.1. Self-monitor learning needs and seek assistance when needed.	✓	✓	✓
D.2. Use study habits necessary to manage academic pursuits and requirements.		✓	✓
D.3. Strive for accuracy and precision.		✓	✓
D.4. Persevere to complete and master tasks.			✓
E.1. Work independently.	✓	✓	✓
E.2. Work collaboratively.	✓	✓	
II. Foundational Skills			
B.1. Write clearly and coherently using standard writing conventions.	✓		✓
B.2. Write a variety of forms for various audiences and purposes.			✓
C.5. Synthesize and organize information effectively.	✓		✓

C.6. Design and present an effective product.			✓
C.8. Present final product.			✓
D.2. Use statistical and probabilistic skills necessary for planning an investigation, and collecting, analyzing, and interpreting data.		✓	✓
D.3. Present analyzed data and communicate finding in a variety of formats.			✓

TCCRS Mathematics Standards Addressed

Performance Expectation	Getting Started	Investigating	Drawing Conclusions
V. Probabilistic Reasoning			
B.1. Computation and interpretation of probabilities.		✓	✓
VIII. Problem Solving and Reasoning			
A.1. Analyze given information	✓	✓	
A.2. Formulate a plan or strategy	✓	✓	
A.3. Determine a solution.			✓
A.4. Justify the solution.			✓
A.5. Evaluate the problem-solving process.			✓
B.1. Develop and evaluate convincing arguments.			✓
IX. Communication and Representation			
B.2. Summarize and interpret mathematical information provided orally, visually, or in written form within the given context.			✓
C.3. Explain, display, or justify mathematical ideas and arguments using precise mathematical language in written or oral communications.			✓

TEKS Standards Addressed

Let the Games Begin - *Texas Essential Knowledge and Skills (TEKS): Math*
111.36.c.1. The student uses a variety of strategies and approaches to solve both routine and non-routine problems. The student is expected to: 111.36.c.1.A. compare and analyze various methods for solving a real-life problem 111.36.c.1.B. use multiple approaches (algebraic, graphical, and geometric methods) to solve problems from a variety of disciplines. 111.36.c.1.C. select a method to solve a problem, defend the method, and justify the reasonableness of the results.
111.36.c.4. The student uses probability models to describe everyday situations involving chance. The student is expected to: 111.36.c.4.A. compare theoretical and empirical probability.

Let the Games Begin

Introduction

Games of probability, or chance, have fascinated people from early times to today. Working in a small design team, you will apply your understanding of probability and expected value to create a game of probability. You will analyze your game and write a proposal to convince a company to invest in your game. Your classmates will play the role of the company and evaluate your game to determine whether your proposal is accepted or denied.

Directions

Getting Started

1. Play the game *Even or Odd?* with a partner for 25 rounds.
2. Complete the *Even or Odd? Game Analysis* handout.
3. Be prepared to share your results regarding outcomes and experimental probability with the class.

Investigating

1. Compute the expected values for both players and compare to your predictions on the *Game Analysis* handout.
2. Consider the fairness of the game. Do you think the game is fair? If so, why? If not, how could the game be changed so that it is fair?

Drawing Conclusions

1. Work in your design team to create the game of probability you will present to the *Carnival-O-Rama* company for funding. Keep in mind the following:
   - The owner of *Carnival-O-Rama* knows that customers are looking for games that are fun, affordable, and have good odds of winning.
   - But, *Carnival-O-Rama* is a business, and while they want their customers to have fun, their ultimate goal is to make a profit.
2. Your game must fulfill the criteria in the *Carnival-O-Rama Game Proposal* handout. You may use any of the materials provided by your instructor or other approved materials such as spinners, number cubes, colored chips, coins, grid paper, and so on.

3. Write a set of directions to explain the rules of your game.
4. Determine how much *Carnival-O-Rama* should charge its customers to play the game, and the approximate value of the prize.
5. Test your game by playing it enough times to be able to make a prediction about what will probably happen. Record any experimental data that would support your proposal to *Carnival-O-Rama*.
6. Determine the theoretical expected value of your game and compare it to the experimental values.
7. Write a proposal detailing your game. Use the *Carnival-O-Rama Game Proposal* handout to ensure your proposal satisfies all criteria listed.
8. Prepare your proposal and present it to your peers. They will listen to your proposal and assess it based on the required criteria.
9. Listen to the presentations of your peers.
10. With your design team, read through the assessments you receive. Reflect on the feedback given. If your team disagrees with an assessment, submit your reasons in writing to your instructor.
11. Organize your work in the following order before handing it in.
    - Rules of game
    - Model of game (if on paper)
    - Experimental data
    - Written proposal
    - Peer assessments
    - Reasons for disagreements with a peer assessment, if needed

Carnival-O-Rama Game Proposal

Your proposal must clearly explain how all the requirements below have been met.

Design Criteria

- The outcome of the game is based on purely random events. No skill is necessary to play the game, and skill will not affect the outcome of the game.
- The rules can be easily understood by the average carnival attendee. The rules include:
  - ⇒ How to play
  - ⇒ What counts as winning
  - ⇒ The cost to play
  - ⇒ What the player wins or the value of the prize
- The model of the game accurately reflects the probabilities presented in the proposal.

Analysis Criteria

- Effectively, clearly, and convincingly communicate the following:
  - ⇒ The number of experimental trials, the tabulated data, and the experimental probability for wins and losses
  - ⇒ Correctly computed theoretical probability with a thorough explanation of how it was derived
  - ⇒ Comparison between experimental probability and theoretical probability and an explanation for any irregularities between the two calculations
  - ⇒ Expected value projected each time the game is played

Other Criteria

- Affordability
- Odds of winning for a potential player
- Long-term profit forecast for *Carnival-O-Rama* from the game

Rules of Play — *Even or Odd?*

This game requires two players.

Players will determine which player will score points with even numbers (the "even player") and which player will score points with odd numbers (the "odd" player).

Players take turns spinning the two spinners.

On a turn, a player spins each spinner once.

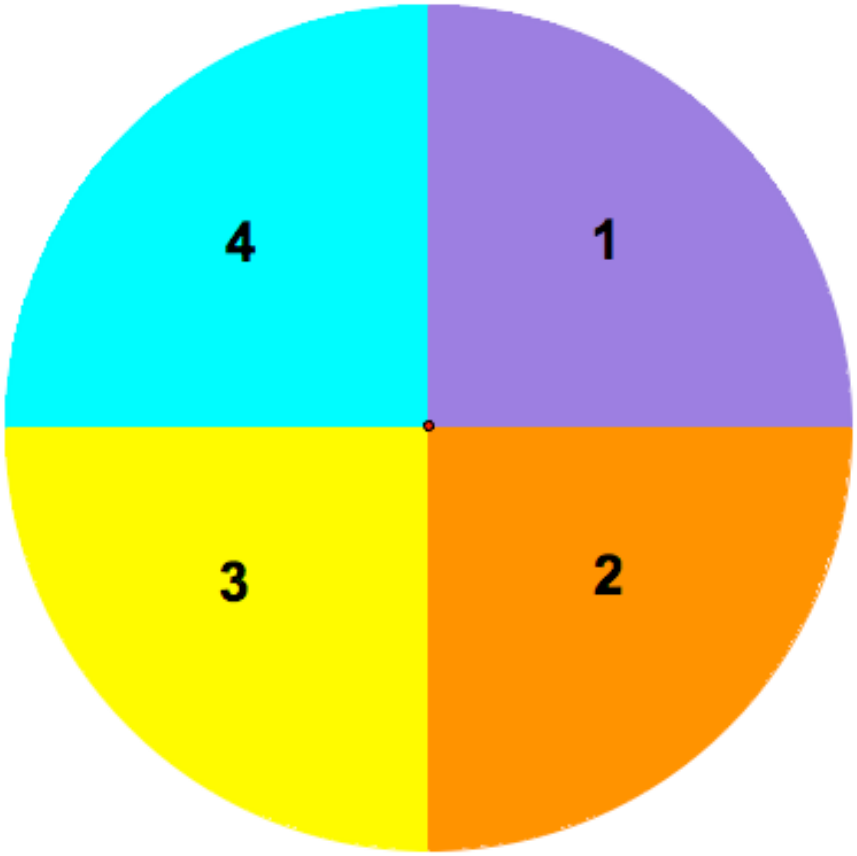
- If the product of the two spins is even, the even player receives 2 points.
- If the product of the two spins is odd, the odd player receives 2 points.

Players keep track of the spinner outcomes and points on the scorecard.

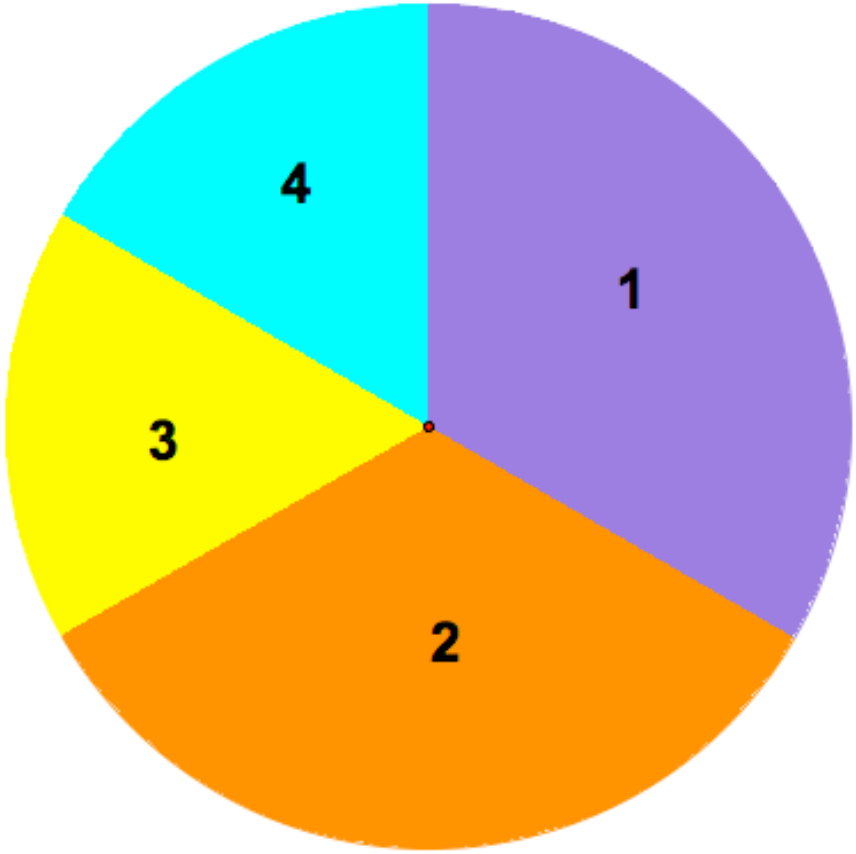
A game consists of 25 "rounds" of turns.

Even or Odd?

Spinner #1



Spinner #2



Score Card — *Even or Odd?*

Even player: _____ Odd player: _____

Round	Spinner #1 Outcome	Spinner #2 Outcome	Product	Points Awarded	
				Player "Even"	Player "Odd"
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					
21					
22					
23					
24					
25					
			Total Points:		

Game Analysis — *Even or Odd?*

After playing 25 rounds of "Even or Odd?" complete the questions below using the data collected on your score sheet.

1. Record the *experimental probability* of obtaining an odd product and an even product. Justify your answer.

   P(odd product) =

   P(even product)=

2. Make an organized list of all of the possible outcomes that might occur in one round using spinner #1 and spinner #2 and the theoretical probability of each.

3. Record the *theoretical probability* of obtaining an odd product and an even product. Justify your answer.

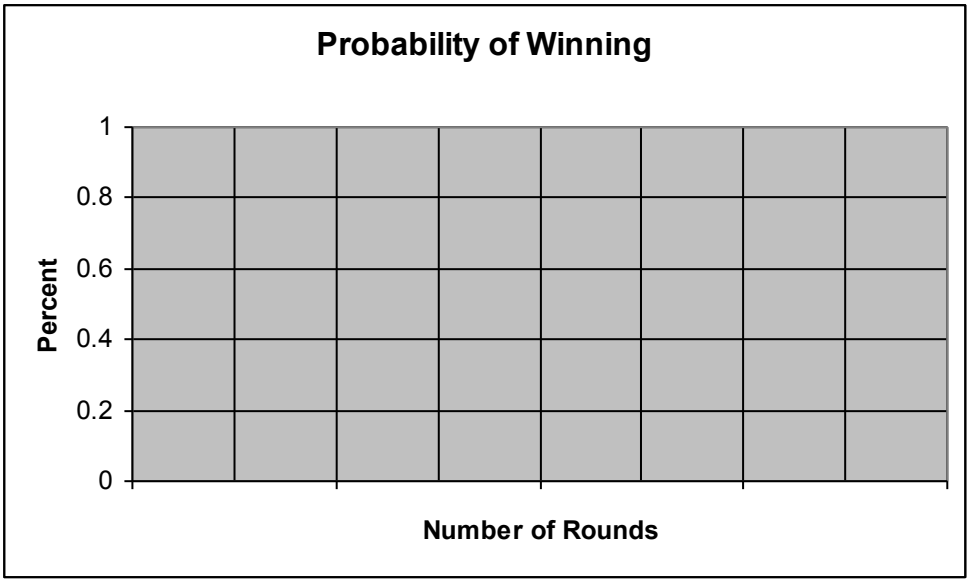
   P(odd product) =

   P(even product)=

4. How do the *experimental* and *theoretical* probabilties compare?

5. Suppose “Even or Odd?” were played 100 times. Predict the number of times you would expect the “even” player to win. How many times would you expect the “odd” player to win? Justify your responses.

6. Using the *experimental* data from your score card, graph the percent of time the “even” player and the “odd” player won after each round of play. Use different colors to represent each player.



7. What would you expect the graph to look like after 50 rounds of play? 100 rounds? 1,000 rounds? Explain.

Carnival-O-Rama Game Proposal Peer Assessment

Game Designers: _____

Assessment provided by: _____

Circle the comment that best describes how the game designers' proposal satisfied the criteria given by *Carnival-O-Rama*. In the shaded area below your assessment, explains your comment. Enter the total number of points in the box above.

1. The rules of the game are complete and easy to understand. The rules clearly explain how to play, how to win, cost to play, and the value of the prize.				
1 - strongly disagree	**2** - disagree	**3** – somewhat agree	**4** - agree	**5** - strongly agree
Feedback:				
2. The game designers created an accurate model of a game involving probability.				
1 - strongly disagree	**2** - disagree	**3** – somewhat agree	**4** - agree	**5** - strongly agree
Feedback:				
3. Carnival attendees would be attracted to this game because it is affordable and the odds of winning appear acceptable.				
1 - strongly disagree	**2** - disagree	**3** – somewhat agree	**4** - agree	**5** - strongly agree
Feedback:				
4. The designers accurately and clearly communicated their experimental probability and how it compared to the theoretical probability. Both odds of winning and losing were included in the proposal.				
1 - strongly disagree	**2** - disagree	**3** – somewhat agree	**4** - agree	**5** - strongly agree
Feedback:				
5. The designers correctly computed the expected profit and gave a reasonable explanation of why the game is expected to produce a profit.				
1 - strongly disagree	**2** - disagree	**3** – somewhat agree	**4** - agree	**5** - strongly agree
Feedback:				